β-perfect graphs that do not necessarily have simplicial extremes

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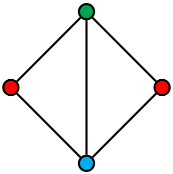
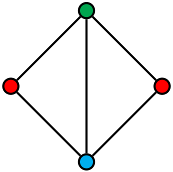
Graph colouring



- A *k-colouring* of a graph is an assigment of k colours to the vertices of the graph such that no two adjacent vertices receive the same colour.

Graph colouring



- A *k-colouring* of a graph is an assigment of k colours to the vertices of the graph such that no two adjacent vertices receive the same colour.
- For a graph G, $\chi(G)$ denotes the minimum number k for which there exists a k-colouring of G. This is called the *chromatic number* of G.

Induced subgraphs and H-freeness

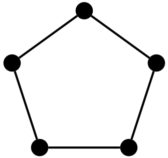


Figure: P_5 is not contained in C_5

- A graph H is an *induced subgraph* of G if H can be obtained from G by deleting vertices (and incident edges).

Induced subgraphs and H-freeness

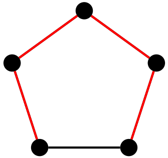


Figure: P_5 is not contained in C_5

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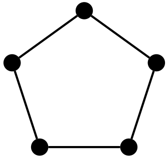
- A graph G *contains* a graph H if some induced subgraph of G is isomorphic to H.

Induced subgraphs and H-freeness



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- G is *H-free* if G does not contain H.

Induced subgraphs and H-freeness

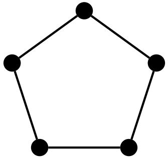
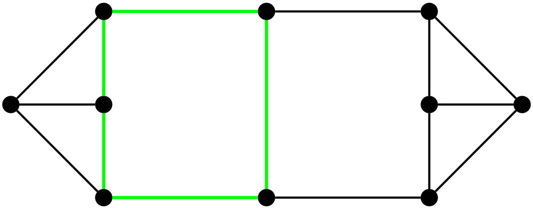


Figure: P_5 is not contained in C_5

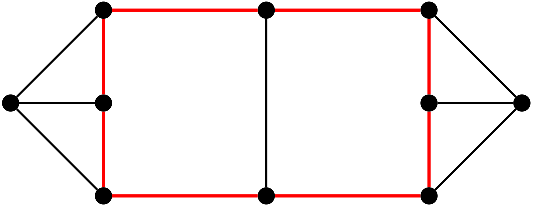
- A graph H is an *induced subgraph* of G if H can be obtained from G by deleting vertices (and incident edges).
- A graph G *contains* a graph H if some induced subgraph of G is isomorphic to H.
- G is *H-free* if G does not contain H.
- G is *$(H_1, \ldots, H_k)$-free* if G is H_i-free for all $i \in \{1, \ldots, k\}$.

Holes



- A *hole* is a chordless cycle of length at least 4.

Holes



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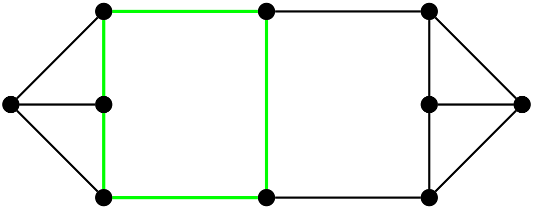

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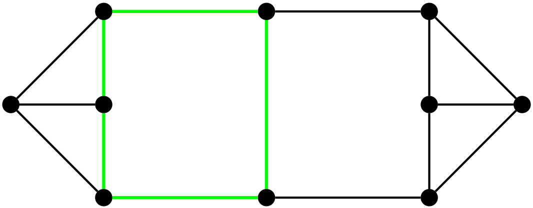

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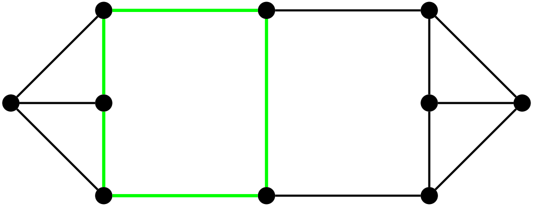
- A hole is *even* or *odd* depending on the parity of its length.

Holes



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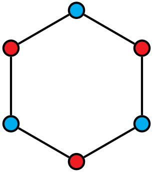
- The *length* of a hole is the number of its vertices.
- A hole is *even* or *odd* depending on the parity of its length.
- Example: even-hole-free $= (C_4, C_6, \ldots)$-free.

β-perfect graphs



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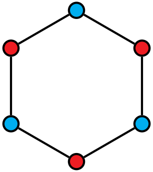
β-perfect graphs



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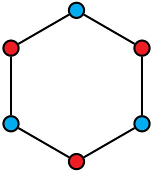
β-perfect graphs



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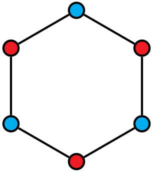
- $\chi(G) \leq \beta(G)$.
- A graph G is *β-perfect* if $\chi(G') = \beta(G')$ for all induced subgraphs G' of G.

β-perfect graphs



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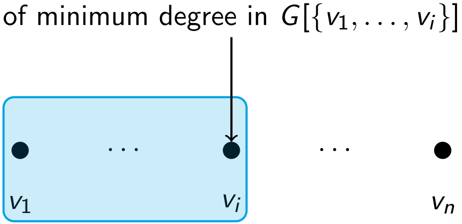
Observation

β-perfect graphs are even-hole-free.

Colouring β-perfect graphs



Observation

The greedy colouring algorithm can optimally colour β-perfect graphs in polynomial time.

What is not known?

Problem

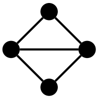
Characterise β-perfect graphs by forbidden induced subgraphs.

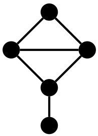
i.e. *G is β-perfect iff it is $(H_1, H_2, \ldots)$-free.*

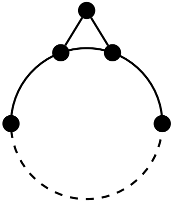
Problem

Can we decide whether a given graph is β-perfect in polynomial time?

Known classes of β-perfect graphs

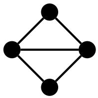

diamond


kite


cap

- Markossian, Gasparian, Reed; 1996 (even hole, diamond, cap)-free

Known classes of β-perfect graphs



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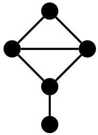

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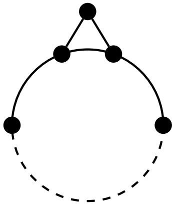

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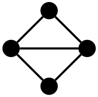
- Markossian, Gasparian, Reed; 1996 (even hole, diamond, cap)-free
- de Figueiredo, Vušković; 2000 (even hole, diamond, cap-on-6-vertices)-free

Known classes of β-perfect graphs



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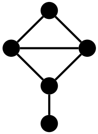

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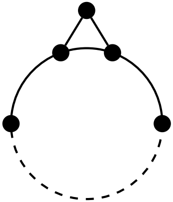

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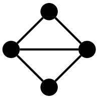

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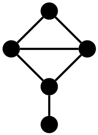

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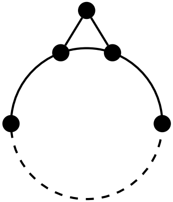

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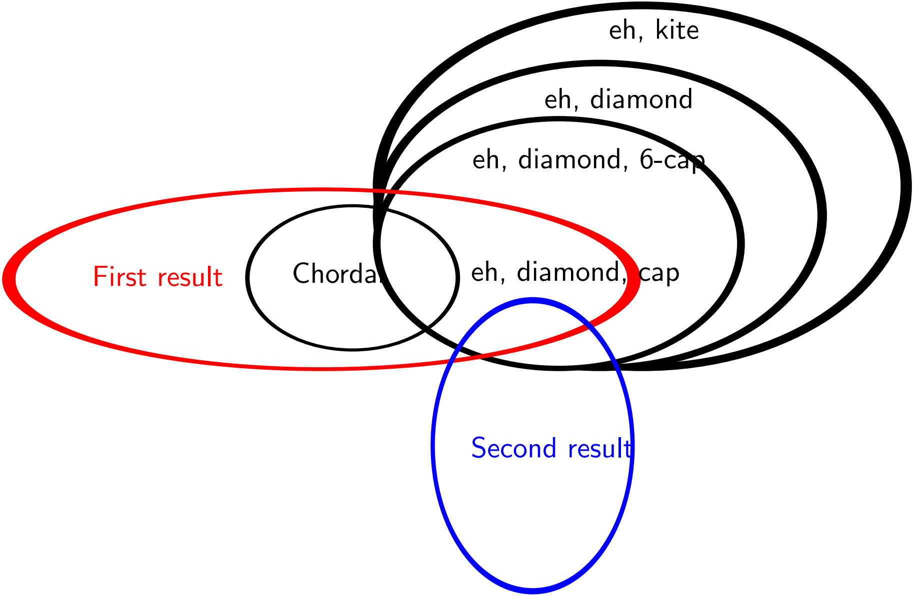
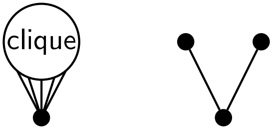
- Fraser, Hamel, Hoàng; 2018 (even hole, kite)-free


eh, kite
eh, diamond
eh, diamond, 6-cap
First result
Chordal
eh, diamond, cap
Second result

Minimal β-imperfect graphs

- A graph is *minimally β-imperfect* if it is not β-perfect but all its proper induced subgraphs are β-perfect.

Minimal β-imperfect graphs



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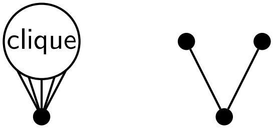
- A vertex is a *simplicial extreme* if its neighbourhood is a clique or of size 2.

Minimal β-imperfect graphs



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Lemma (Markossian, Gasparian, Reed; 1996)

If G is a minimally β-imperfect graph that is not an even hole, then G has no simplicial extreme.

β-perfection via simplicial extremes

Theorem (Dirac; 1961)

Every chordal graph contains a vertex whose neigbourhood is a clique.

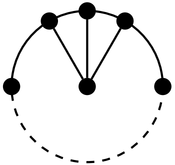
β-perfection via simplicial extremes

Theorem (Dirac; 1961)

Every chordal graph contains a vertex whose neigbourhood is a clique.

- So chordal graphs are β-perfect.
- In fact, all mentioned results were proved by showing the existence of simplicial extremes.

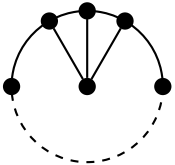
Graphs without simplicial extremes



twin wheel

Let $\mathcal{C}$ be the class of (even hole, twin wheel, cap)-free graphs.

Graphs without simplicial extremes



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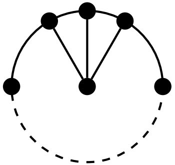
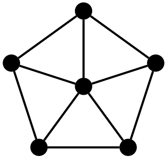
- Goal: prove that every graph in $\mathcal{C}$ is β-perfect.

Graphs without simplicial extremes



twin wheel



A graph in $\mathcal{C}$ with no simplicial extreme.

Let $\mathcal{C}$ be the class of (even hole, twin wheel, cap)-free graphs.

- Goal: prove that every graph in $\mathcal{C}$ is β-perfect.
- Issue: there are graphs in $\mathcal{C}$ that have no simplicial extreme.

A decomposition theorem

Theorem (Dirac; 1961)

If G is a chordal graph, then G either:

- *is a complete graph, or*
- *has a clique cutset.*

A decomposition theorem

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Theorem (Cameron, da Silva, Huang, Vušković; 2018)

If G is (even hole, cap)-free, has a hole, and has no clique cutset, then G is obtained from an (even hole, $\triangle$)-free graph with no clique cutset by:

- *blowing up vertices into cliques, and*
- *adding a universal clique.*

A decomposition theorem

Theorem

If G is an (even hole, twin wheel, cap)-free graph, then:

- *G is a complete graph, or*
- *G has a clique cutset, or*
- *G consists of an (even hole, $\triangle$)-free graph that has a hole but no clique cutset, together with a universal clique.*

β-perfection of complete graphs

Theorem

If G is an (even hole, twin wheel, cap)-free graph, then:

- *G is a complete graph, or*
- *G has a clique cutset, or*
- *G consists of a triangle-free graph that has a hole but no clique cutset, together with a universal clique.*

- Complete graphs are chordal and hence β-perfect.

β-perfection of certain $\triangle$-free graphs

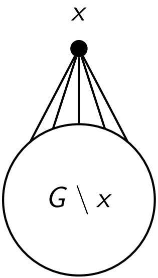
Theorem

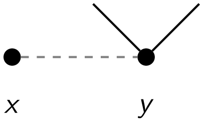
If G is an (even hole, twin wheel, cap)-free graph, then:

- *G is a complete graph, or*
- *G has a clique cutset, or*
- *G consists of a triangle-free graph that has a hole but no clique cutset, together with a universal clique.*

- Partition (A, B) of $V(G)$ so that $G[A]$ is an (even hole, triangle)-free graph, and $G[B]$ is a complete graph.
- $G[A]$ is (even hole, diamond, cap)-free, and hence is β-perfect.
- Adding a universal clique preserves β-perfection.

A useful lemma



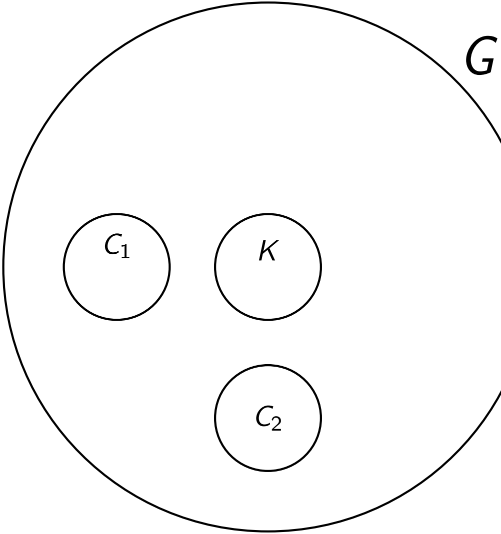


Lemma (Markossian, Gasparian, Reed; 1996)

Let G be an (even hole, triangle)-free graph.
For every vertex $x \in V(G)$, either:

- *x is universal, or*
- *there is a vertex y nonadjacent to x with $d(y) \leq 2$.*

Finishing up



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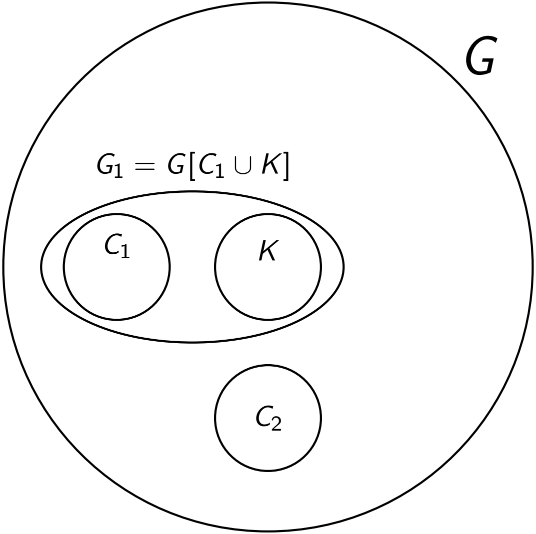

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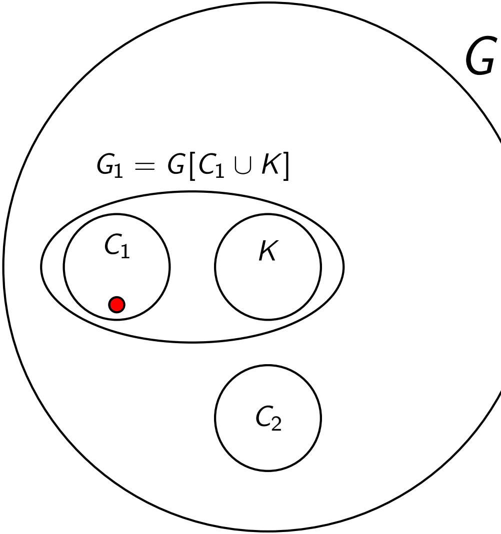

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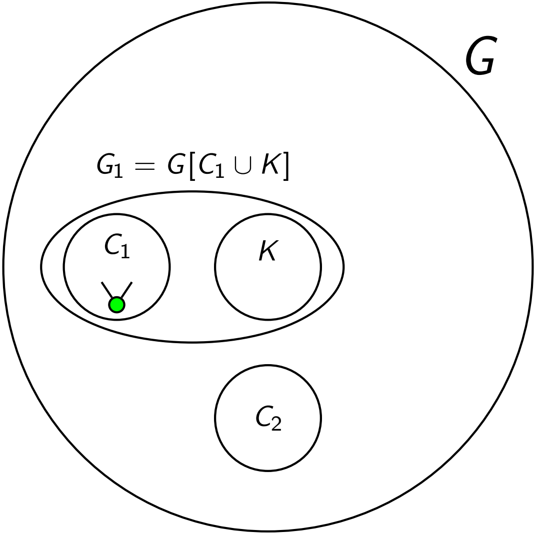
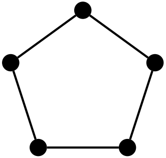


$$\beta(G) = \delta(G) + 1 \leq d(x) + 1 = \chi(G_1) \leq \chi(G)$$

Summary of the first part

- (even hole, twin wheel, cap)-free graphs are β-perfect
- These graphs do not necessarily contain simplicial extremes.
- We find simplicial extremes in the basic graphs. After decomposing, we look at what happens to these simplicial extremes as we 'go back upwards' to the original graph.

Hyperholes



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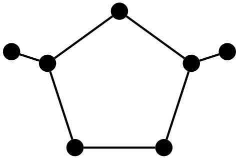
Hyperholes



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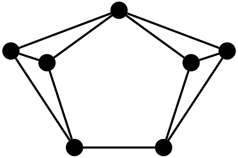

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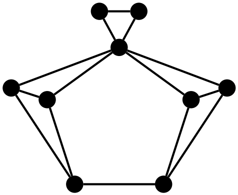

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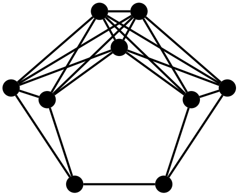

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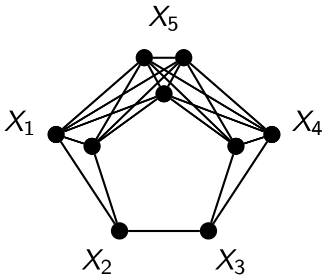

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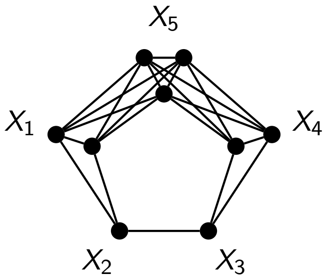
Hyperholes



- A *k-hyperhole* is any graph obtained from a hole of length k by clique substitutions.
- We write $H = (X_1, \ldots, X_k)$.
- $X_1, \ldots, X_k$ are the *bags* of H.

Facts about hyperholes

- $H = (X_1, \ldots, X_k)$ a k-hyperhole. we assume k is odd
- $\chi(H) = \max\left\{\omega(H), \left\lceil \frac{|V(H)|}{\alpha(H)} \right\rceil\right\}$

Facts about hyperholes

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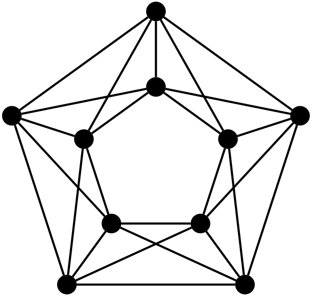
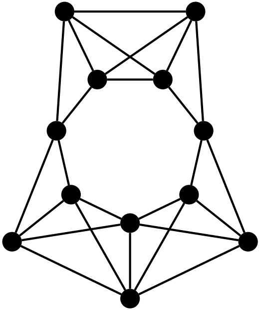
- $\omega(H) = \max\{|X_i \cup X_{i+1}| \colon 1 \leq i \leq k-1\}$

The 5-hyperholes and 7-hyperholes

Here are the only minimally β-imperfect 5-hyperholes and 7-hyperholes.





Theorem

Let $H = (X_1, \ldots, X_k)$ *be a hyperhole with* $k \in \{5, 7\}$*.*
Then H *is* β*-perfect if and only if for some* $i \in \{1, \ldots, k\}$*:*

- $(k = 5) \quad |X_i| = 1$;

$$\cdots \quad \boxed{1} \quad \cdots$$

- $(k = 7) \quad |X_i| = |X_{i+1}| = 1$ *or* $|X_i| = |X_{i+2}| = 1$.

$$\cdots \quad \boxed{1} \quad \boxed{1} \quad \cdots \qquad \text{or} \qquad \cdots \quad \boxed{1} \quad \boxed{} \quad \boxed{1} \quad \cdots$$

Odd hyperholes of length at least 9

"Trivial" hyperholes:

- Three consecutive bags of size one

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- A super-sector containing only 2-sectors

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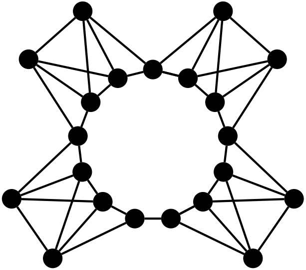
- One 0-sector, all other sectors are of length 2



β-perfection of trivial hyperholes

Lemma

If a hyperhole $H = (X_1, \ldots, X_k)$ contains:

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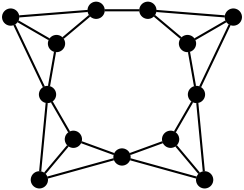
Corollary

Trivial hyperholes are β-perfect.

Base hyperholes



A hyperhole is a *base hyperhole* if

- no three consecutive bags are of size 1;
- every bag has size at most 2;
- no two consecutive bags are of size 2.

Why base hyperholes?

Lemma

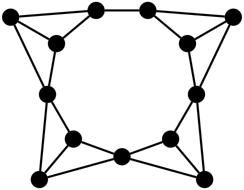
Every nontrivial hyperhole contains a base hyperhole.

Proof sketch.

Apply certain reduction rules to ‘sectors’ and ‘super-sectors’ of the hyperhole. The result is a base hyperhole. □

So let’s characterise β-perfect base hyperholes.

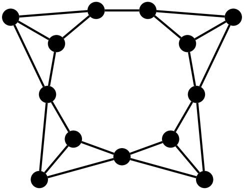
Base hyperholes



For a base hyperhole H of length k:

- $\beta(H) = 4$ and $\omega(H) = 3$;

Base hyperholes



For a base hyperhole H of length k:

- $\beta(H) = 4$ and $\omega(H) = 3$;
- every proper induced subgraph is either chordal, or has three consecutive bags of size 1 $\Longrightarrow$ β-perfect;

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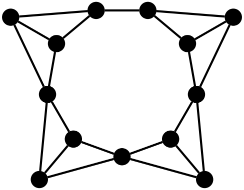

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- H is β-perfect $\Longleftrightarrow$ $\chi(H) = 4$;

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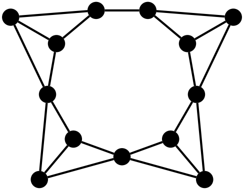

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- every proper induced subgraph is either chordal, or has three consecutive bags of size 1 $\Longrightarrow$ β-perfect;
- H is β-perfect $\iff \chi(H) = 4$;
- H is minimally β-imperfect $\iff \chi(H) = 3$.

Base hyperholes with $\chi = 3$

- $\chi(H) = \max\{\omega(H), \left\lceil \frac{2|V(H)|}{k-1} \right\rceil\}$

Base hyperholes with $\chi = 3$

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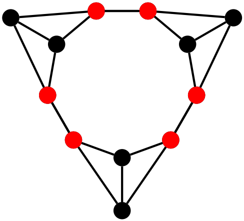

- At least two pairs of consecutive bags of size 1.

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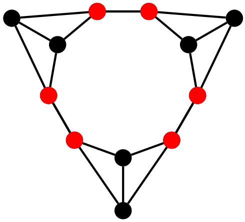

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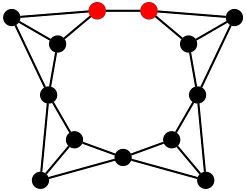

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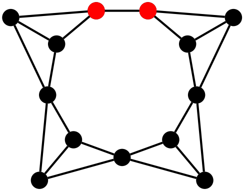


- Exactly one pair of consecutive bags of size 1.
- We say that such a base hyperhole is *good*.

Characterisation of β-perfect base hyperholes

Lemma

A base hyperhole is β-perfect (length odd and at least 9)

if and only if

it is good.

Extending this to nontrivial hyperholes

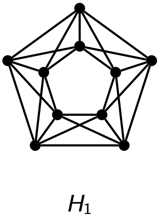
- Start with a *good* base hyperhole.
- We may only add vertices to *specific bags* in order to preserve β-perfection.
- Anything else creates a bad base hyperhole $\Longrightarrow$ not β-perfect

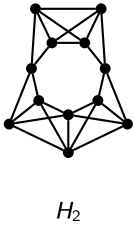
Lemma

If H is a β-perfect hyperhole of length at least 9, then it is:

- *a trivial hyperhole, or*
- *an "extension" of a good base hyperhole.*

The characterisation





Theorem

A hyperhole is β-perfect if and only if it contains no

- *even hole*
- *bad base hyperhole*
- H_1
- H_2

as an induced subgraph.

What next?

- In a hyperhole, we insist that consecutive bags are pairwise complete. What if we relax this condition?

- Can we generalise the result that (even hole, twin wheel, cap)-free graphs are β-perfect?

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thank you for listening!