β-PERFECT GRAPHS

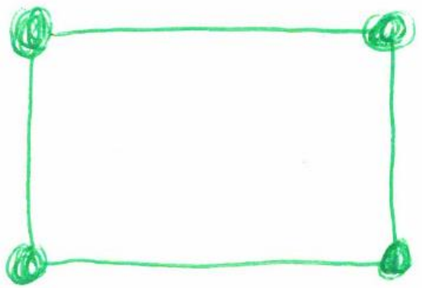
JAKE HORSFIELD

$\beta(G) = \max\{\delta(G')+1 \mid G' \text{ is an induced subgraph of } G\}$

$\chi(G) \leq \beta(G)$

MARKOSSIAN, GASPARIAN, REED (1996):

G is β-PERFECT if $\chi(G') = \beta(G')$
for all induced subgraphs G' of G.



even hole

$\chi = 2$
$\beta = 3$

$\Downarrow$

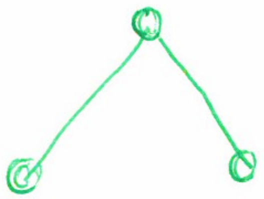
β-perfect graphs are even-hole-free

$v_1, v_2, \ldots, v_i, \ldots, v_n$

v_i is of minimum degree in $G[\{v_1, v_2, \ldots, v_i\}]$.

The greedy colouring algorithm applied to this ordering produces an optimal colouring in polynomial time for β-perfect graphs.

Vertex v is a simplicial extreme if



degree ≤ 2

$N(v)$ clique

THM (MGR)

G minimally β-imperfect + G is not an even hole

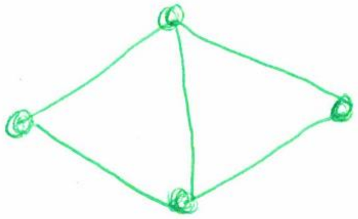
$\Rightarrow$ G contains no simplicial extreme.

To prove that a class of graphs $\mathcal{C}$ is β-perfect, we can show that every graph in $\mathcal{C}$ contains a simplicial extreme.

THM (Dirac 1961)
Every chordal graph has a simplicial vertex

$\Longrightarrow$ chordal graphs are β-perfect.

even hole



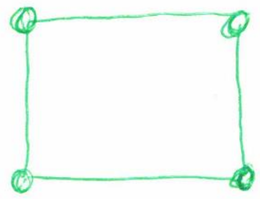
diamond

cap

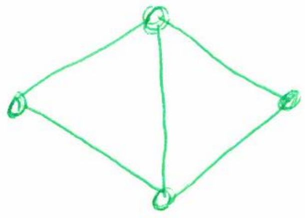
THM (MGR)

G (even hole, diamond, cap)-free $\Longrightarrow$ G β-perfect.

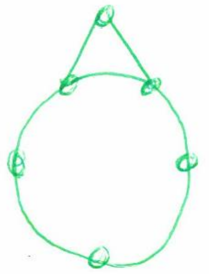
(they show that every (even hole, diamond, cap)-f graph contains a simplicial extreme)



even hole



diamond

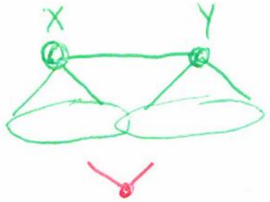


cap-on-6

THM (de Figueiredo, Vušković 2000)

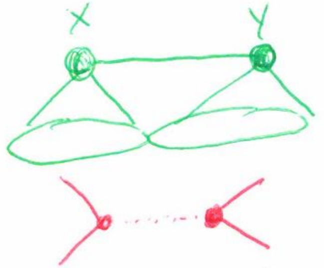
G (even hole, diamond, cap-on-6)-free $\Longrightarrow$

(i) G is chordal, or



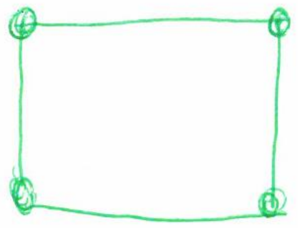
(ii) G contains a C_5, and $\forall xy \in E(G)$, G has a simplicial extreme in $G \setminus (N(x) \cup N(y))$, or

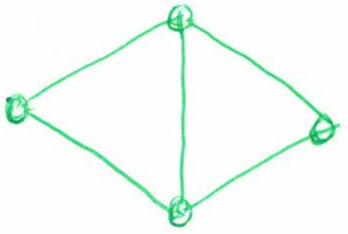
(iii) G contains no C_5, and $\forall xy \in E(G)$, G has two nonadjacent simplicial extremes in $G \setminus (N(x) \cup N(y))$

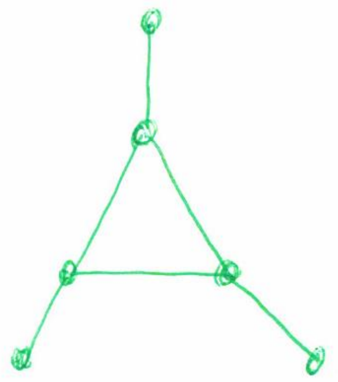


CONJECTURE (dFV)

Excluding only even holes and diamonds is sufficient for β-perfectness.


even hole


diamond


net

THM (KEIJSPER, TEWES 2002)

G (even hole, diamond, net)-free $\Longrightarrow$ G β-perfect.

They also look at β-perfect line graphs.

Established some connections between β-perfectness and regularity.

THM (KT)

G β-perfect $\Rightarrow$ G contains no regular induced subgraphs except perhaps odd holes and cliques.

THM (KT)

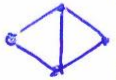
G 3-regular connected even-hole-free not K_4 $\Rightarrow$ G is minimally β-imperfect.

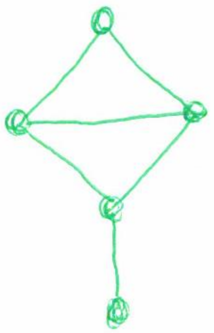
Kloks, Müller, Vušković obtained a decomposition theorem for (even hole, diamond)-free graphs, which led to the following.

THM (KMV 2009)

G (even hole, diamond)-free

$\Longrightarrow$ G contains a simplicial extreme.

In particular, G (even hole, )-free $\Longrightarrow$ G clique, or
G contains two nonadj. simplicial extremes.



Kite

N(v) has ≤ 1 non-edge

near-simplicial vertex

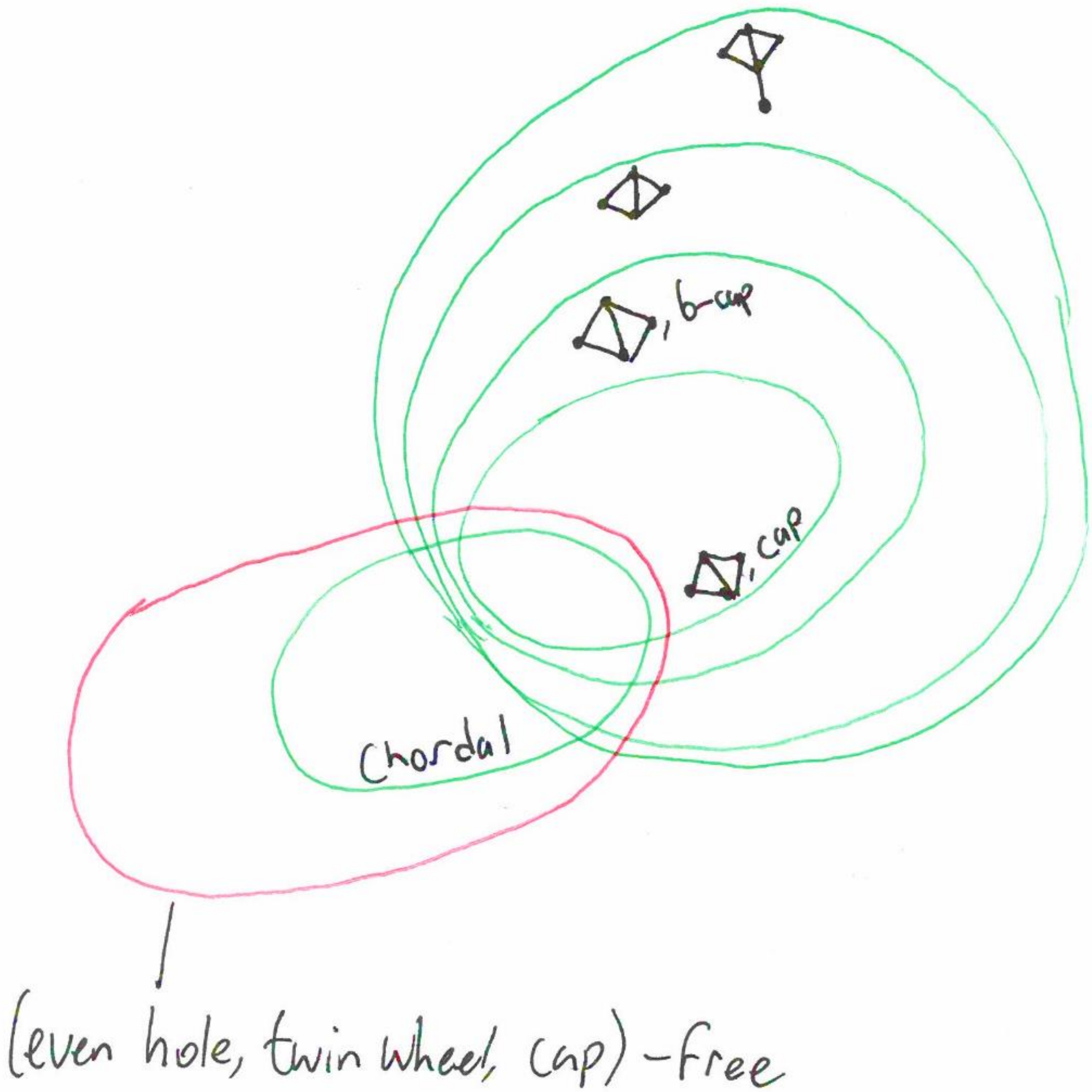
THM (Fraser, Hamel, Hoàng 2018)

G (even hole, kite)-free $\Longrightarrow$

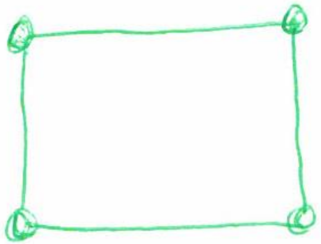
(i) G clique, or

(ii) G has 2 nonadj. near-simplicial vertices.

$\Longrightarrow$ (evenhole, kite)-free graphs are β-perfect.



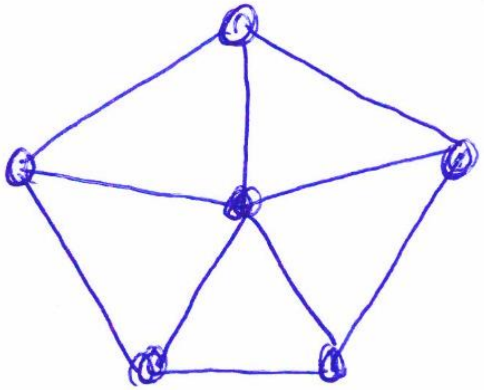
, 6-cap
, cap
Chordal
(even hole, twin wheel, cap)-free



even hole

twin wheel

cap



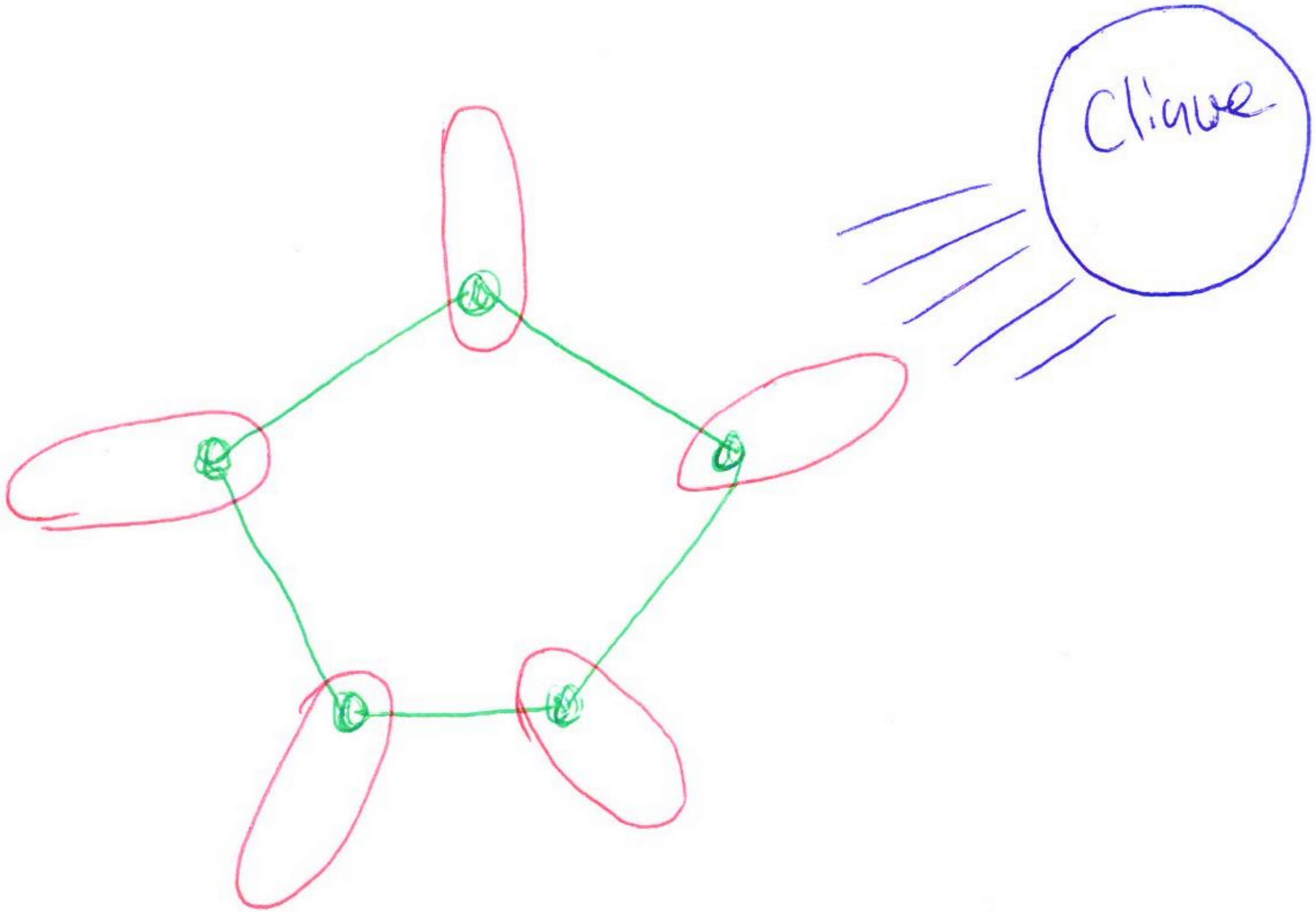
An (even hole, twin wheel, cap)-free graph with no simplicial extreme!

THM (Dirac 1961)

G Chordal $\Rightarrow$ G is Complete, or
G has a Clique cutset.

THM (Cameron, da Silva, Huang, Vušković 2018)

G (even hole, cap)-free, has a hole, and has no Clique cutset
$\Rightarrow$ G obtained from a (even hole, Δ)-free graph with no Clique cutset by blowing up vertices into cliques and adding a universal clique.


Clique

THM

G (even hole, twin wheel, cap)-free $\Longrightarrow$

(i) G complete, or 

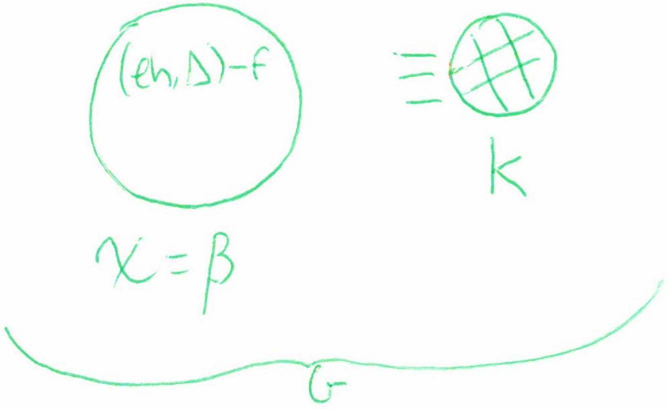
(ii) G consists of a Δ-free graph on at least 3 vertices with a hole and no clique cutset, together with a (possibly empty) universal clique, or

(iii) G has a clique cutset.

β-PERFECTNESS OF THE BASIC GRAPHS

(i) G complete $\Rightarrow$ G chordal $\Rightarrow$ G β-perfect.

(ii) G (even hole, Δ)-free $\Rightarrow$ G (even hole, diamond, cap)-free
+ universal clique
$\Rightarrow$ G β-perfect



$$\chi(G) = \chi + |K| = \beta + |K| = \beta(G)$$

So G has a clique cutset

$\Rightarrow$ G has an extreme clique cutset, say K.

$\forall$ connected components C_i of $G \setminus K$,

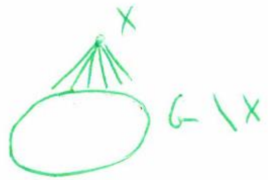
Set $G_i = G[V(C_i) \cup K]$

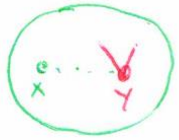
THM (MGR 1996)

Let G be an (even hole, Δ)-free graph.

$\forall x \in V(G)$. x is universal, or

$\exists y \in G \setminus N[x]$. such that $d(y) \leq 2$.





Let G be a minimally β-imperfect graph that is (even hole, twin wheel, cap)-free.

That is, G is <u>not</u> β-perfect, but all proper induced subgraphs of G <u>are</u> β-perfect.

Then $\beta(G) = \delta(G) + 1$.

OPEN PROBLEMS

RECOGNITION

Is there a polynomial time algorithm for recognising β-perfect graphs?

FORBIDDEN INDUCED SUBGRAPH CHARACTERIZATION

G β-perfect iff $(\ldots)$-free