Special 2-joins

Jake Horsfield

Based on joint work with:
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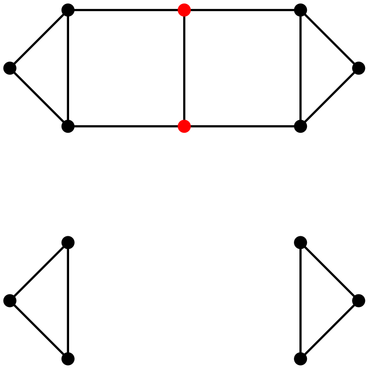
Nicolas Trotignon Kristina Vušković

Cutsets

A *cutset* is any set of vertices or edges whose removal disconnects the graph.



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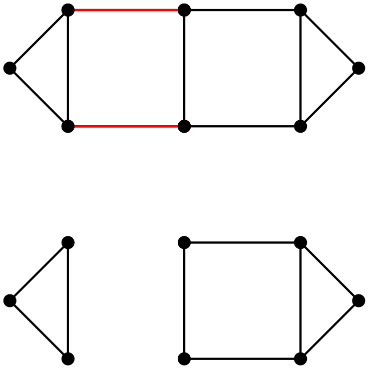
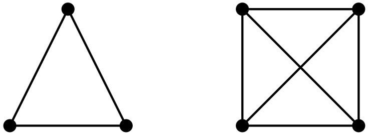
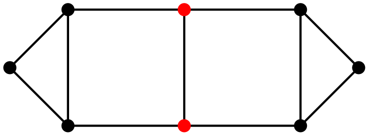


Clique cutsets

A *clique* is a set of pairwise adjacent vertices.

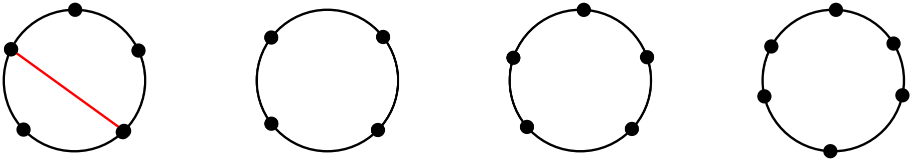


A *clique cutset* is a cutset that is a clique.



Holes

A *hole* is a chordless cycle of length at least 4.



The class $\mathcal{C}_k$

$\mathcal{C}_k =$ the class of graphs G such that every hole of G is of length k.

Goal

Every graph in $\mathcal{C}_k$ is either “basic” or has a clique cutset.

So the question is: **what are the “basic” graphs?**

The class $\mathcal{C}_k$

$\mathcal{C}_k =$ the class of graphs G such that every hole of G is of length k.

Goal

Every graph in $\mathcal{C}_k$ is either "basic" or has a clique cutset, *for odd* $k \geq 7$.

- So from now on, k is assumed to be odd and at least 7.

So the question is: **what are the "basic" graphs?**

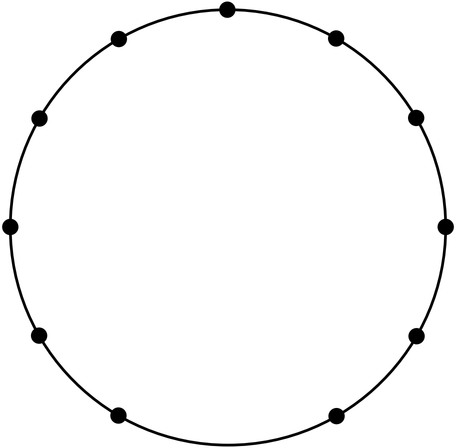
The class $\mathcal{C}_k$

Contains:

- Chordal graphs (they contain no holes)
- Hole of length k

$\mathcal{C}_k$ is a subclass of even-hole-free graphs.

Rings: a generalisation of a hole



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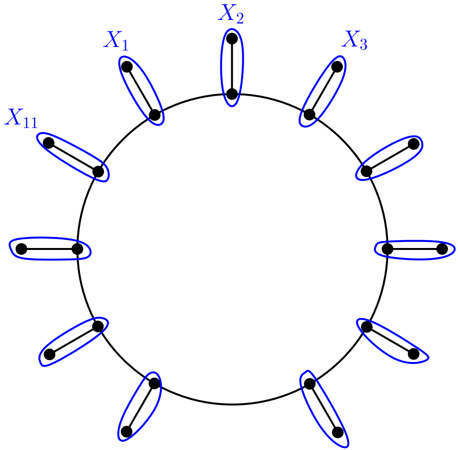

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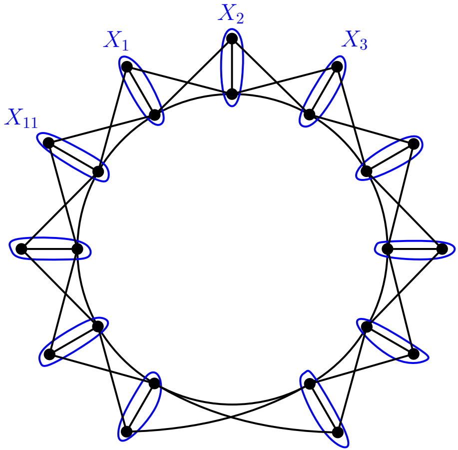

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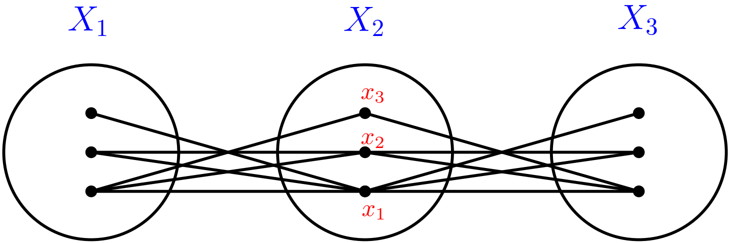
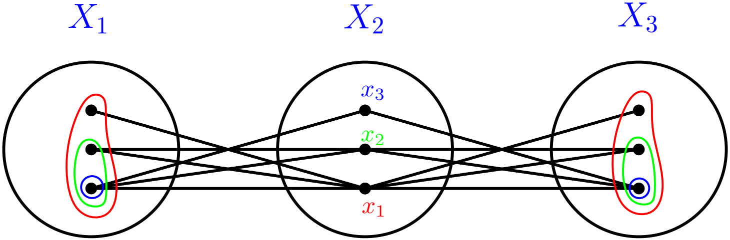
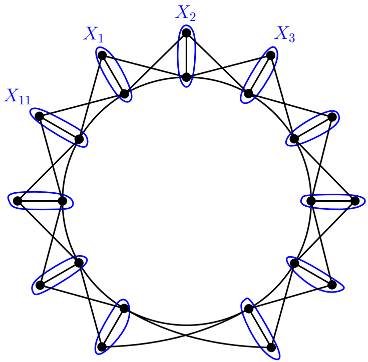


Rings: a generalisation of a hole



Rings

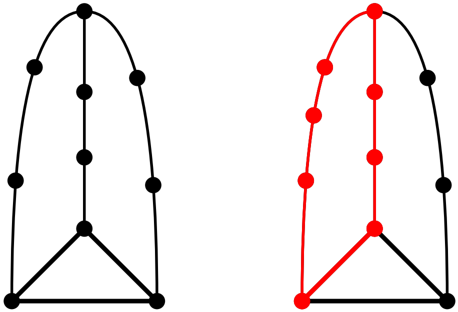


Fact

If H is a hole in a ring of length k, then H is of length k.

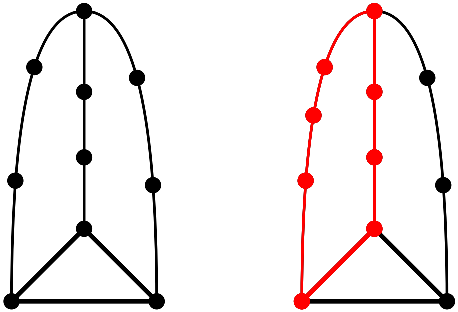
So the rings of length k belong to $\mathcal{C}_k$.

Pyramids



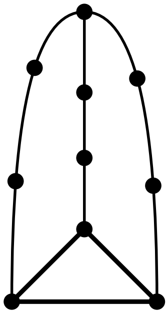
- Pyramids whose 3 paths are of the same length belong to $\mathcal{C}_k$, for some k.

Pyramids



- Pyramids whose 3 paths are of the same length belong to $\mathcal{C}_k$, for some k.
- All holes in such a pyramid are of odd length.

Generalisations of pyramids



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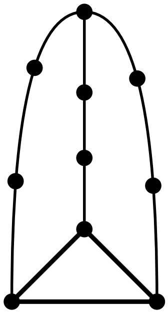
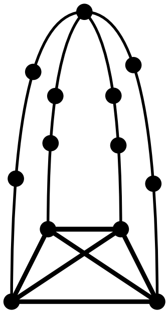
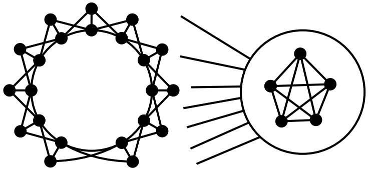




Graphs containing no pyramid

As a consequence of a result of Boncompagni, Penev and Vušković:

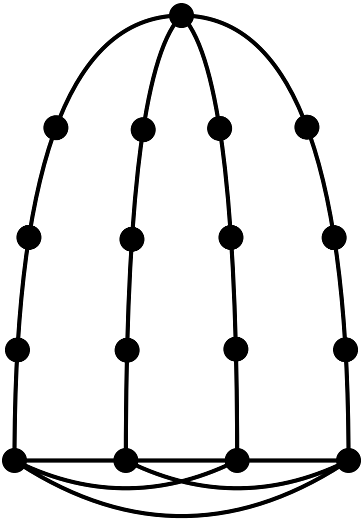


Lemma

If $G \in \mathcal{C}_k$ and G contains no pyramid, then

- G is an odd ring together with a universal clique, or
- G has a clique cutset.

Generalising pyramids



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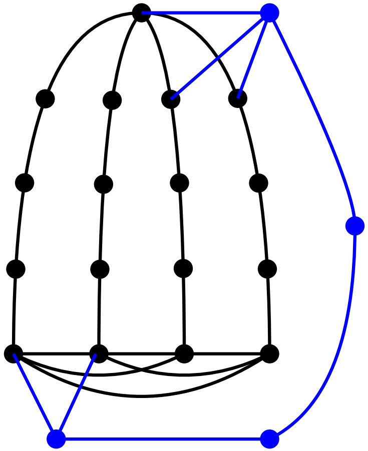

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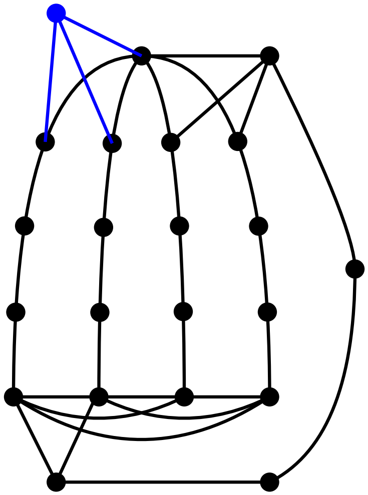

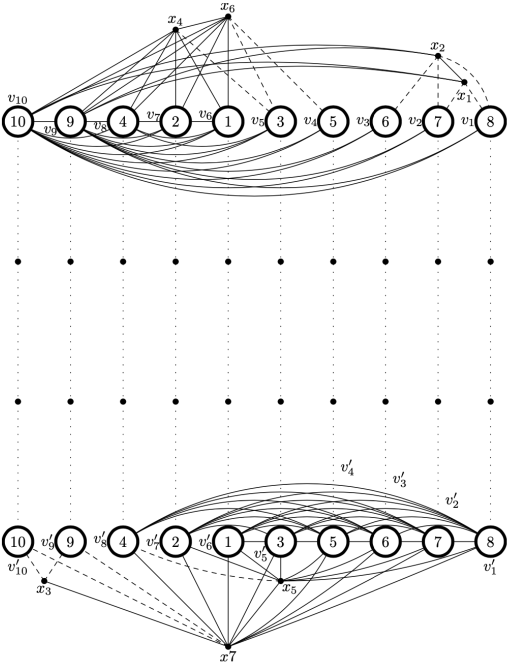
x_6
x_4
x_2
x_1
v_{10}
10
9
4
2
1
3
5
6
7
8
v_9
v_8
v_7
v_6
v_5
v_4
v_3
v_2
v_1
v'_4
v'_3
v'_2
10
v'_9
9
v'_8
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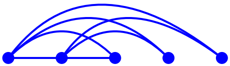
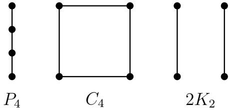
v'_7
2
v'_6
1
3
5
6
7
8
v'_5
v'_{10}
v'_1
x_3
x_5
$x7$

Templates



1. Build a threshold graph with vertex set A.



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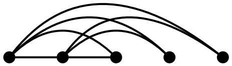

1. Build a threshold graph with vertex set A.
2. Take the complement of $G[A]$; call its vertex set A'.

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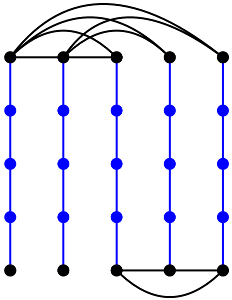

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3. Connect each vertex of A to its corresponding vertex in A' with a path of length $\ell \geq 2$.

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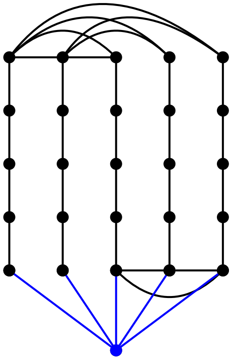

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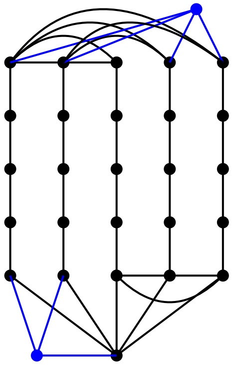

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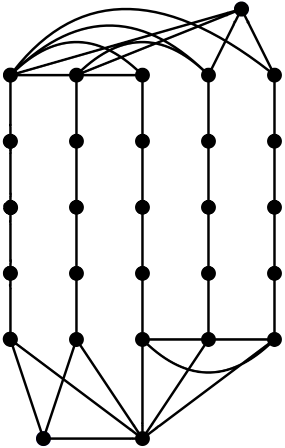
4. If some vertex of $G[A]$ (resp. $G[A']$) is isolated, then add a vertex that is complete to A (resp. A').
5. Possibly add some more vertices by considering a certain type of hypergraph on A (resp. A').

Blowing up templates



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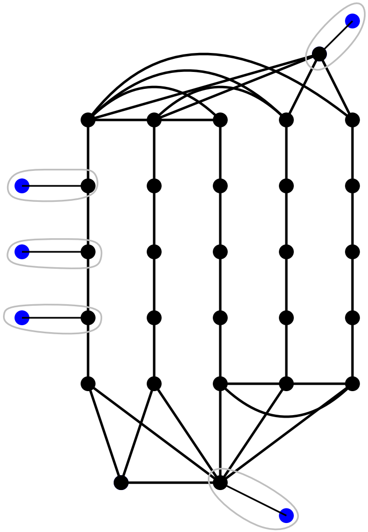

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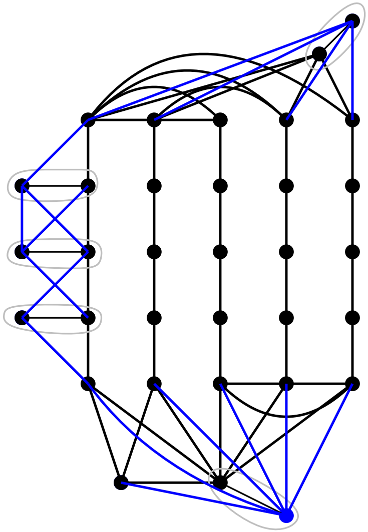

Decomposition theorem

Theorem

For every odd $k \geq 7$, every graph G in $\mathcal{C}_k$ is

- *a ring,*
- *or a blowup of a template,*

or has

- *a universal vertex*
- *or a clique cutset.*

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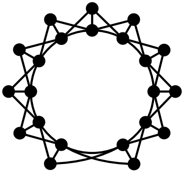

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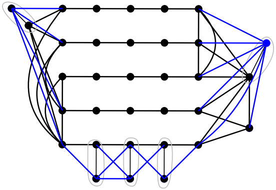

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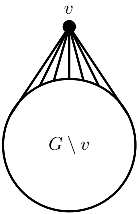

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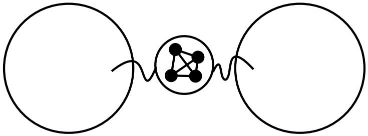

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The recognition problem

Problem

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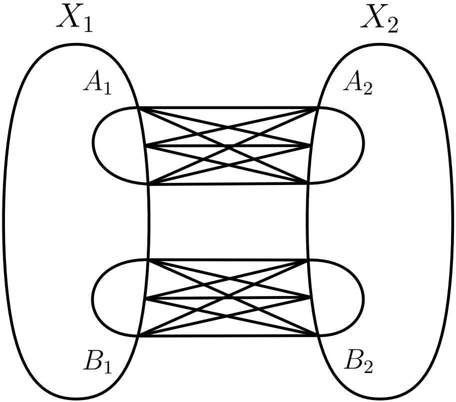
We use "special" 2-joins to obtain a more efficient algorithm.

2-joins

An edge cutset introduced by Cornuéjols and Cunningham in 1985.



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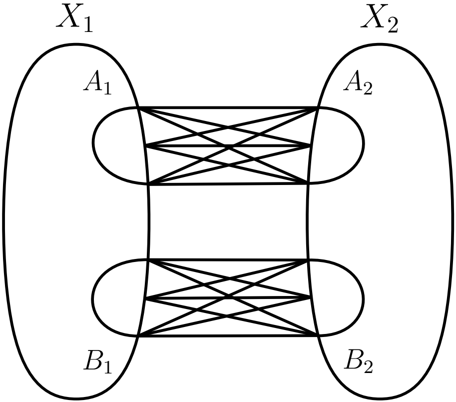

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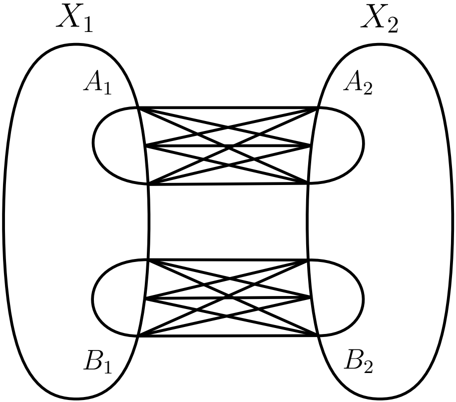

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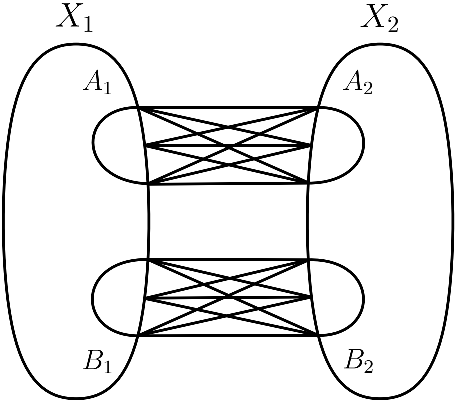

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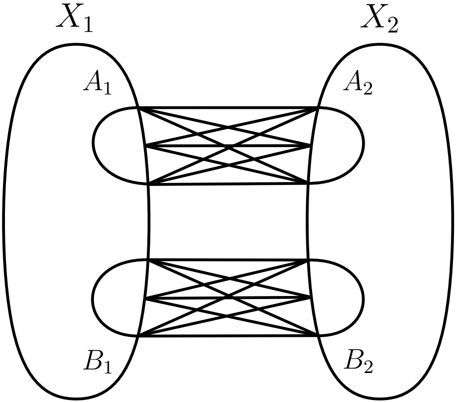

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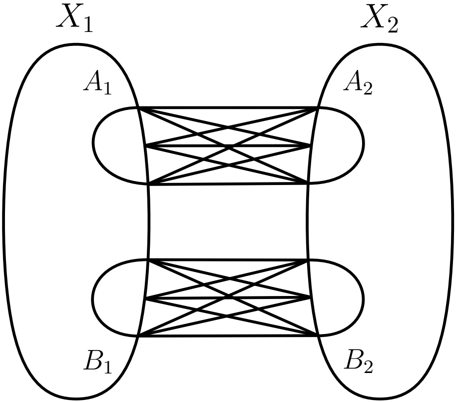

- (X_1, X_2) partition of $V(G)$
- A_1, A_2, B_1, B_2 nonempty and pairwise disjoint
- A_1 is complete to A_2
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- There are no other edges between X_1 and X_2

2-joins

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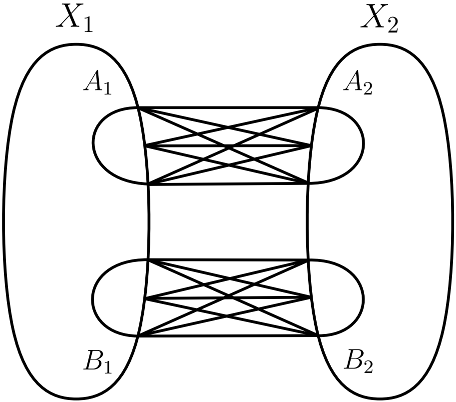
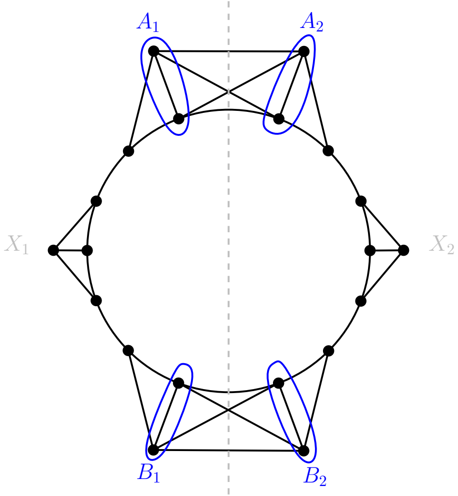
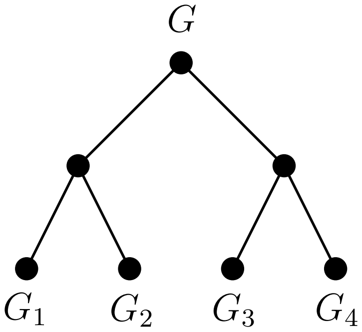


- (X_1, X_2) partition of $V(G)$
- A_1, A_2, B_1, B_2 nonempty and pairwise disjoint
- A_1 is complete to A_2
- B_1 is complete to B_2
- There are no other edges between X_1 and X_2
- ... couple more conditions

2-joins example



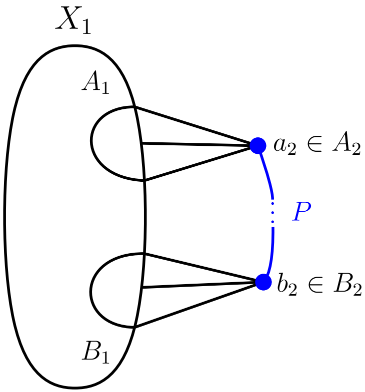
Decomposition tree



Goal

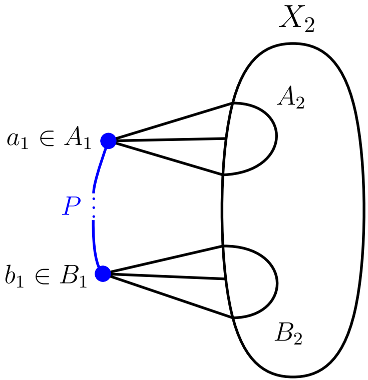
G has property P if and only if each of the leaves $(G_1, \ldots, G_4)$ has property P.

Blocks of decomposition



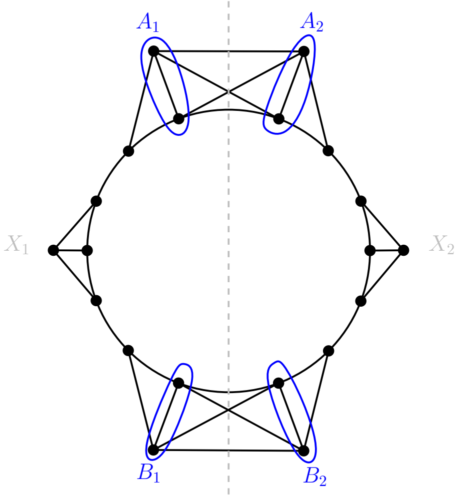
$$G_1 = G[X_1 \cup P]$$

Blocks of decomposition

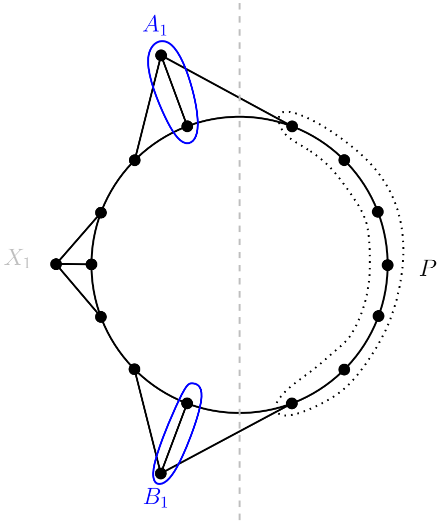


$$G_2 = G[X_2 \cup P]$$

Blocks of decomposition example



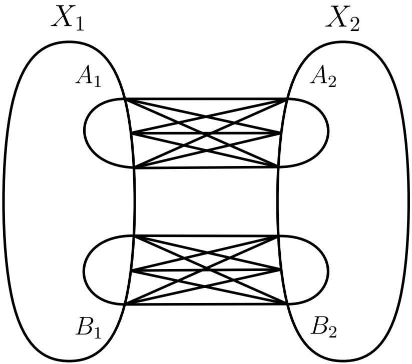
Blocks of decomposition example



The problem with “normal” 2-joins

- Not always class-preserving (a graph may not belong to $\mathcal{C}_k$ but its blocks of decomposition do)
- Not all templates have 2-joins – but they have a “2-join-like” decomposition.

Special 2-joins of type 1



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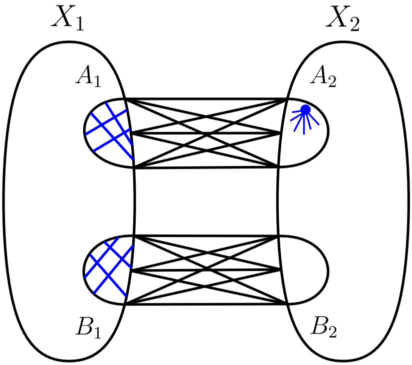
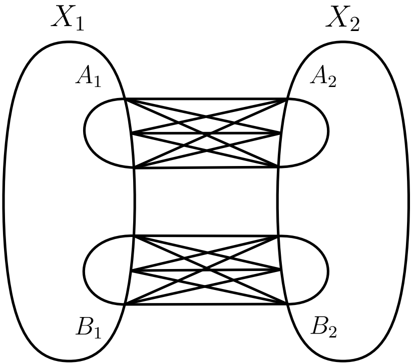


- A_1 and B_1 are cliques
- At least one of $G[A_2]$ and $G[B_2]$ contains a universal vertex

Special 2-joins of type 2



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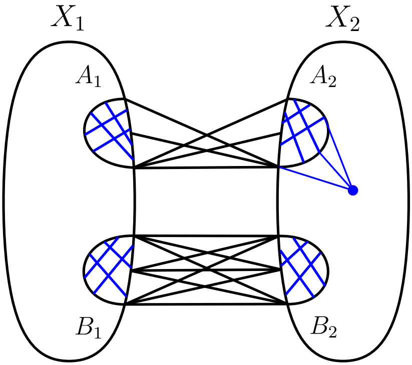
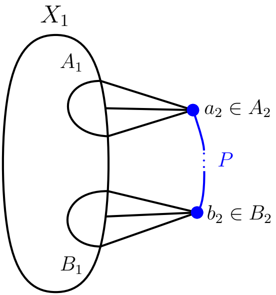


- A_1, A_2, B_1 and B_2 are cliques
- A_1 and A_2 are nested
- B_1 is complete to B_2
- Some vertex of $X_2 \setminus A_2$ is complete to A_2

Special 2-joins are useful

Lemma

Let G be a graph and let (X_1, X_2) be a special 2-join (of type 1 or 2) of G. Let G_1 and G_2 be the blocks of decomposition of G w.r.t. (X_1, X_2). Then $G \in \mathcal{C}_k$ if and only if $G_1, G_2 \in \mathcal{C}_k$ (for all $k \geq 5$).



$G_1 = G[X_1 \cup P]$

Decomposition theorem II

Theorem

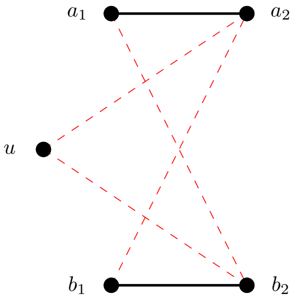
For every odd $k \geq 7$, every graph G in $\mathcal{C}_k$ is:

- *a ring*
- *or a pyramid,*

or has

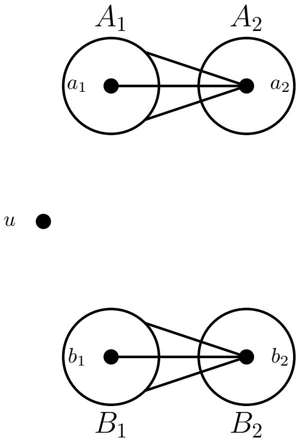
- *a universal vertex,*
- *a clique cutset,*
- *or a special 2-join (of type 1 or 2).*

Detecting special 2-joins of type 1



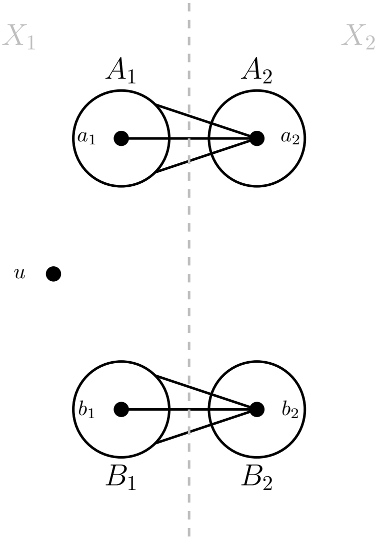
- Guess vertices a_1, a_2, b_1, b_2 and u.

Detecting special 2-joins of type 1



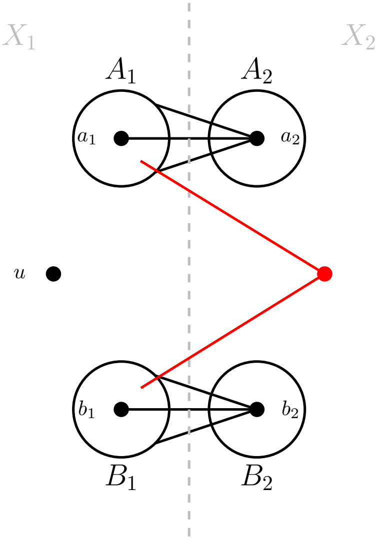
- Set $A_1 = N(a_2)$ and $B_1 = N(b_2)$.
- Set $A_2 = \{x \in X_2 : N(x) \cap A_1 \neq \emptyset\}$
- Set $B_2 = \{x \in X_2 : N(x) \cap B_1 \neq \emptyset\}$

Detecting special 2-joins of type 1



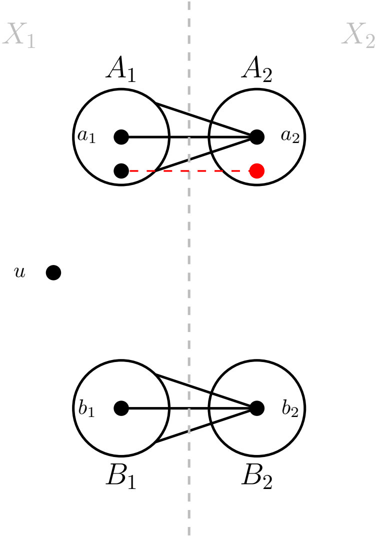
If there is a special 2-join of type 1 with $a_1, b_1, u \in X_1$ and $a_2, b_2 \in X_2$, then the following rules *must* be applied.

Detecting special 2-joins of type 1



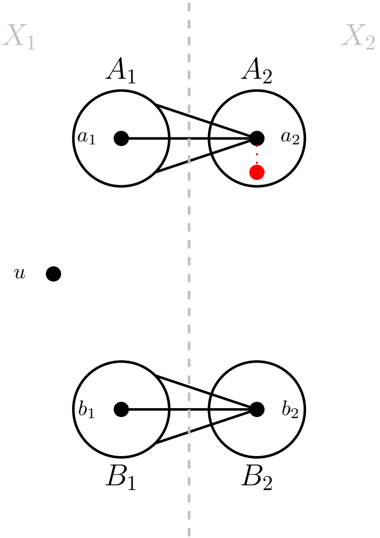
If $x \in X_2$ has neighbours in both A_1 and B_1, then move x to X_1.

Detecting special 2-joins of type 1



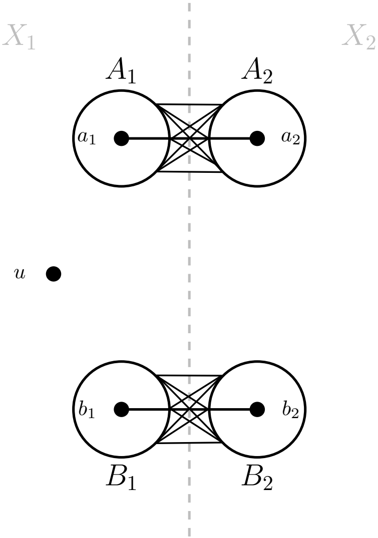
If $x \in A_2$ but is not complete to A_1, then move x to X_1.

Detecting special 2-joins of type 1



If $x \in A_2 \setminus \{a_2\}$ and is nonadjacent to a_2, then move x to X_1.

Detecting special 2-joins of type 1



Once no more rules can be applied, check whether A_1 and B_1 are cliques + some other check.

Time complexity

Problem

Given a graph G, decide whether $G \in \mathcal{C}_k$ for some odd $k \geq 7$.

Something like $\mathcal{O}(n^8)$ — exact details in the works!

Summary

- $\mathcal{C}_k$ = the class of graphs G such that every hole of G is of length k.
- A decomposition theorem for graphs in $\mathcal{C}_k$, for odd $k \geq 7$.
- A decomposition-based recognition algorithm for this class using two variations on 2-joins.

Summary

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thanks for listening 🙂